A Lego-Brick Approach to Lossy Source Coding

Nadim Ghaddar

University of California San Diego

Collaborators: Shouvik Ganguly & Lele Wang & Young-Han Kim

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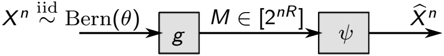
Lossy Source Coding Problem

- Setup:
  - Compression of a source with some less-than-perfect fidelity
  - Encoder g, decoder ψ, rate R, distortion level D
  - Problem: Given (n, R, θ, D), design (g, ψ) s.t. $\frac{1}{n}\,\mathsf{E}[d_H(X^n, \widehat{X}^n)] \leq D$



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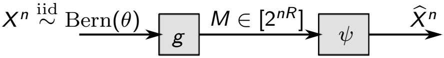

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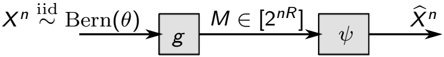

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- Coding schemes based on point-to-point channel codes:
  - Trellis-based quantizers [Viterbi-Omura'74]
  - LDPC codes with large CN degrees and optimal encoding [Matsunaga-Yamamoto'03]
  - LDGM codes with message-passing encoding [Wainwright-Maneva'05]
  - Polar codes [Korada-Urbanke'10]
  - $\cdots$

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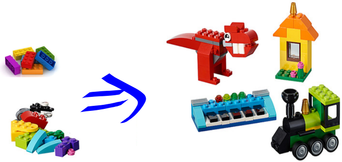

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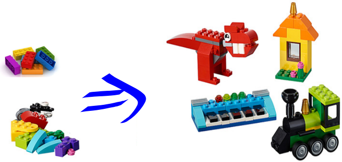
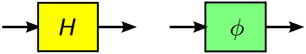


For a given coding problem,

- What "Lego bricks" to assemble, and what properties should they satisfy?
- How to assemble Lego bricks?
- How do performance guarantees translate?

Lego Bricks

- Basic Lego Bricks:



  - P2P code (H, ϕ) for symmetric DMC
  - Parity-check matrix H, decoder ϕ
  - Dimension k, blocklength n
  - Probability of error ϵ
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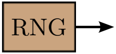
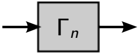


  - Shared random dither



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- Notation: WLOG, let $H = \begin{bmatrix} A & B \end{bmatrix}$ where B is nonsingular, and denote

$$\widetilde{H} = \begin{bmatrix} \mathbf{0} \\ B^{-1}H \end{bmatrix}.$$

Coding Scheme

- Lossy source coding problem:
  - Let $X^n \overset{\text{iid}}{\sim} \text{Bern}(\theta)$.
  - Goal: Compression of X^n with expected distortion $D \in [0, \theta]$.
- Approach: Target a conditional distribution $p(x \,|\, \hat{x}) \sim \text{BSC}(D)$.

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$$q(x, v \,|\, \hat{x}) \triangleq p_{X,\widehat{X}}(\hat{x} \oplus v, x)$$

s.t. for $X^n \overset{\text{iid}}{\sim} \text{Bern}(\theta)$, $V^n \overset{\text{iid}}{\sim} \text{Bern}(1/2)$ and $U^n = \phi_2\left(X^n, V^n\right) \oplus V^n$, we have

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- Note:
  - Channel q is symmetric with $\pi(x, v) = (x, v \oplus 1)$.

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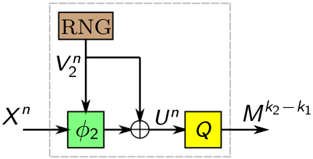

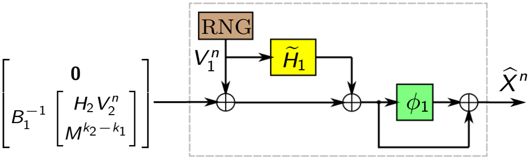

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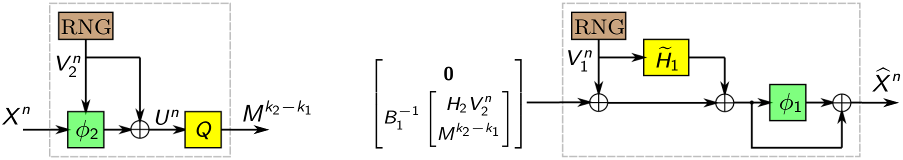

- Comments:
  - V_2^n is a random dither shared between encoder and decoder.
  - Sequence U^n satisfies

$$d_{TV}(p_{X^n,U^n}, \prod p_{X,\widehat{X}}) \leq \delta$$

  - $H_1 = \begin{bmatrix} A_1 & B_1 \end{bmatrix}$, where B_1 is nonsingular.
  - Rate of coding scheme is $R = \frac{k_2 - k_1}{n}$.
  - Expected distortion is

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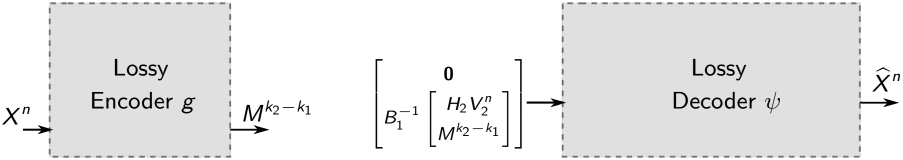

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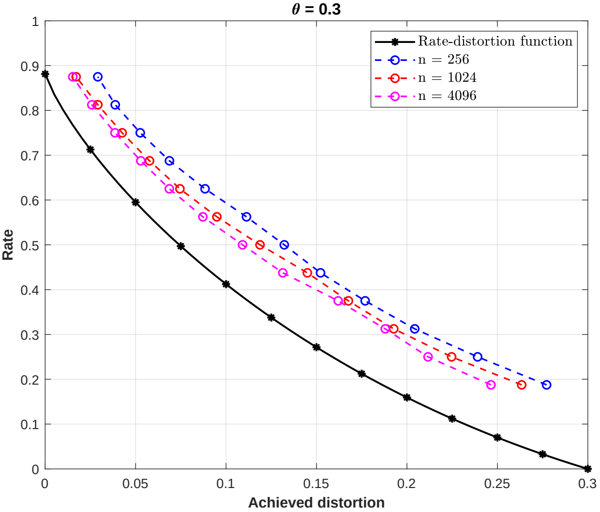

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2. Avoiding nestedness:
   - An alternative construction allows to avoid the nestedness condition
   - Less implementation-friendly
   - Has a block-Markov structure (inputs to one coding block depend on previous blocks)

Beyond Lossy Source Coding: Coding over Networks

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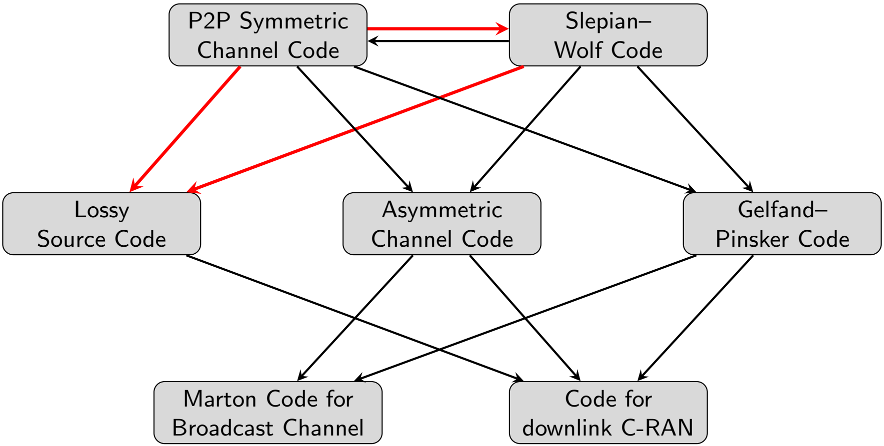

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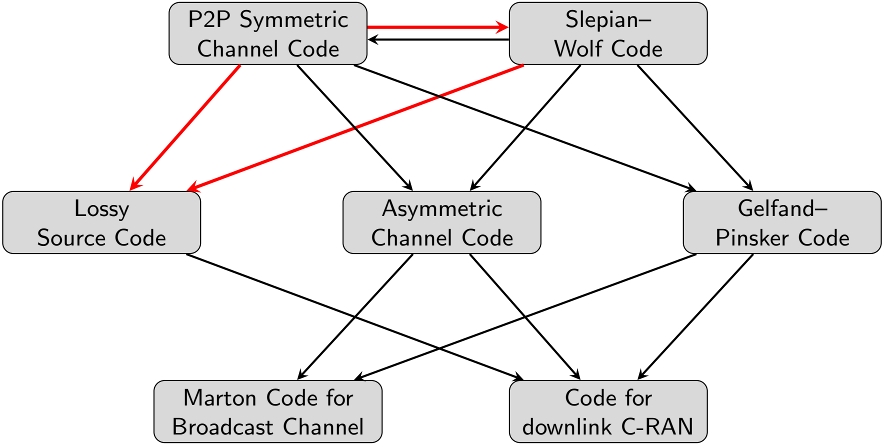

- All coding schemes can be constructed starting from P2P symmetric channel codes.
- All constructions are rate-optimal if the constituent Lego bricks are rate-optimal.