Sensing Codebook Design for Structured Sparse Recovery Problems

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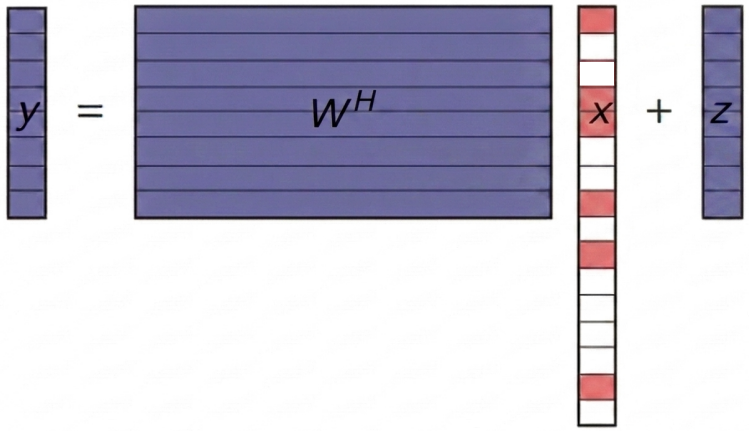
[2]University of Toronto, Canada

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Halifax, Canada

June 8, 2026

Compressed Sensing Problem

General setting:

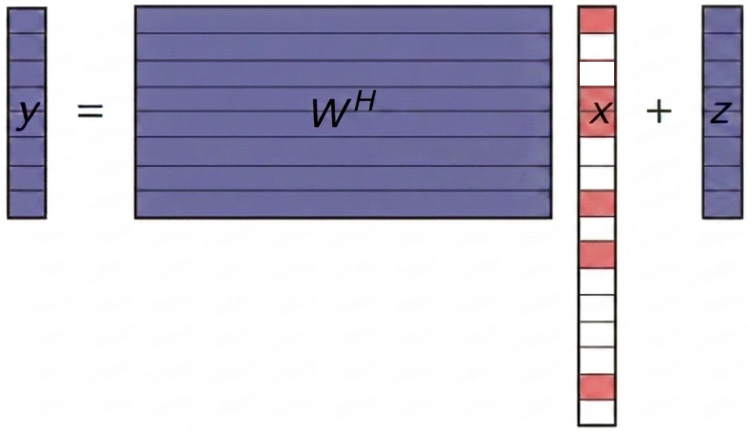


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- Take $T \ll N$ linear measurements using a sensing matrix $\mathbf{W}$
- Gaussian noise $\mathbf{z} \in \mathbb{C}^T$

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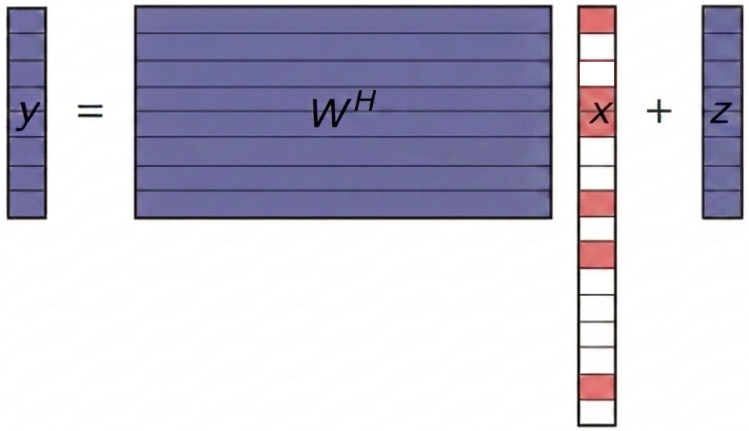
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- **Applications:** Wireless communications/sensing, medical imaging, seismology, etc.

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- In many instances, $\mathbf{x}$ may not be sparse, but has a sparse representation:

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where $\mathbf{u} \in \mathbb{C}^M$ is sparse and $\mathbf{D} \in \mathbb{C}^{N \times M}$ is a given dictionary matrix.

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- Example: AoA estimation (later...)

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- This presentation: Focus on 1-sparse compressed sensing

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   - Sample $N \times T$ matrix $\mathbf{W}$ with i.i.d. entries (e.g. Gaussian)

     $\Rightarrow \mathbf{W}$ satisfies the restricted isometry property (RIP) with high probability

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     $\Rightarrow$ Inspiration: Channel coding...

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Connection to Channel Coding

- Main focus: 1-sparse compressed sensing

$$\mathbf{y} = \alpha \mathbf{W}^{\mathsf{H}} \mathbf{D} \mathbf{u} + \mathbf{z}$$

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- Key connections to channel coding:
  1. $\mathbf{W}^{\mathsf{H}} \mathbf{D} \mathbf{u}$ can be interpreted as a codeword that encodes the "message" $\mathbf{u}$
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- Key distinctions from channel coding:
  1. Sensing codewords take the form $\{\mathbf{W}^{\mathsf{H}} \mathbf{d}_i\}_{i=1}^M$
  2. Unit norm constraint is typically imposed on the columns of $\mathbf{W}$ (not the codewords)

Problem Formulation

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- Two settings:
  1. Known α (or estimated): Maximize minimum ℓ_2 distance between sensing codewords

$$\underset{\mathbf{W}}{\text{maximize}} \quad \min_{i \neq j} \left\| \mathbf{W}^{\mathsf{H}} \mathbf{d}_i - \mathbf{W}^{\mathsf{H}} \mathbf{d}_j \right\|_2$$

     $\Rightarrow$ Minimum ℓ_2 distance dictates coding performance when α is known
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$$\underset{\mathbf{W}}{\text{maximize}} \quad \min_{i \neq j} \sqrt{1 - \frac{|\langle \mathbf{W}^{\mathsf{H}} \mathbf{d}_i, \mathbf{W}^{\mathsf{H}} \mathbf{d}_j \rangle|^2}{\|\mathbf{W}^{\mathsf{H}} \mathbf{d}_i\|^2 \|\mathbf{W}^{\mathsf{H}} \mathbf{d}_j\|^2}}$$

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- **Main contribution:** Solve the two optimizations via a convex iteration method

Setting 1: Known α

- Maximize minimum ℓ_2 distance:

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   where $\mathbf{\Gamma}_{\mathbf{W}} = \mathrm{diag}(\|\mathbf{w}_1\|^2, \ldots, \|\mathbf{w}_T\|^2)$

2. Writing $\mathbf{R} = \mathbf{W} \mathbf{\Gamma}_{\mathbf{W}}^{-1} \mathbf{W}^{\mathsf{H}}$, and using epigraph formulation,

$$\begin{aligned} \underset{\mathbf{R}, \gamma}{\text{maximize}} \quad & \gamma \\ \text{subject to} \quad & (\mathbf{d}_i - \mathbf{d}_j)^{\mathsf{H}} \mathbf{R} (\mathbf{d}_i - \mathbf{d}_j) \geq \gamma, \qquad \forall\, i \neq j, \\ & \mathrm{tr}(\mathbf{R}) = T, \quad \mathbf{R} \succeq 0, \\ & \mathrm{rank}(\mathbf{R}) \leq T \end{aligned}$$

   $\Rightarrow$ Rank-constrained SDP

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3. Fix γ: Search for matrix $\mathbf{R}$ with squared minimum distance γ

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$\Rightarrow$ Iterate between solving these two problems for $I_{\max}$ iterations

[3] Jon Dattorro. *Convex Optimization & Euclidean Distance Geometry*. Meboo Publishing, 2005.

Setting 1: Known α

3. Fix γ: Search for matrix $\mathbf{R}$ with squared minimum distance γ

$$\begin{aligned} \text{find} \quad & \mathbf{R} \\ \text{such that} \quad & (\mathbf{d}_i - \mathbf{d}_j)^{\mathsf{H}} \mathbf{R} (\mathbf{d}_i - \mathbf{d}_j) \geq \gamma, \qquad \forall\, i \neq j, \\ & \operatorname{tr}(\mathbf{R}) = T, \quad \mathbf{R} \succeq 0, \\ & \operatorname{rank}(\mathbf{R}) \leq T \end{aligned}$$

$\Rightarrow$ Rank-constrained feasibility problem

4. **Key observation:** Solution of least rank must be an extreme point of the feasible set

$\Rightarrow \exists$ a hyperplane that supports the feasible set at that extreme point

$\Rightarrow$ Search for hyperplane via a convex iteration method[3]:

$$\begin{cases} \min\limits_{\mathbf{R}} & \operatorname{tr}(\mathbf{R}\mathbf{Q}^*) \\ \text{s.t.} & (\mathbf{d}_i - \mathbf{d}_j)^{\mathsf{H}} \mathbf{R} (\mathbf{d}_i - \mathbf{d}_j) \geq \gamma, \ \forall\, i \neq j, \\ & \operatorname{tr}(\mathbf{R}) = T, \quad \mathbf{R} \succeq 0 \end{cases} \underset{\mathbf{Q}^*}{\overset{\mathbf{R}^*}{\rightleftarrows}} \begin{cases} \min\limits_{\mathbf{Q}} & \operatorname{tr}(\mathbf{R}^*\mathbf{Q}) \\ \text{s.t.} & \mathbf{0} \preceq \mathbf{Q} \preceq \mathbf{I}, \\ & \operatorname{tr}(\mathbf{Q}) = N - T \end{cases}$$

$\Rightarrow$ Iterate between solving these two problems for $I_{\max}$ iterations

$\Rightarrow$ If optimal value converges to 0, rank-T solution is found $\Rightarrow$ Increase γ

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Setting 2: Unknown α

- Maximize minimum chordal distance:

$$\begin{aligned} \underset{\mathbf{W}}{\text{maximize}} \quad & \min_{i \neq j} \sqrt{1 - \frac{|\langle \mathbf{W}^{\mathsf{H}} \mathbf{d}_i, \mathbf{W}^{\mathsf{H}} \mathbf{d}_j \rangle|^2}{\|\mathbf{W}^{\mathsf{H}} \mathbf{d}_i\|^2 \|\mathbf{W}^{\mathsf{H}} \mathbf{d}_j\|^2}} \\ \text{subject to} \quad & \|\mathbf{w}_t\| = 1, \quad \forall\, t \end{aligned}$$

1. Equivalently,

$$\underset{\mathbf{W}}{\text{minimize}} \quad \max_{i \neq j} \frac{\left|\langle \mathbf{\Gamma}_{\mathbf{W}}^{-1/2} \mathbf{W}^{\mathsf{H}} \mathbf{d}_i, \mathbf{\Gamma}_{\mathbf{W}}^{-1/2} \mathbf{W}^{\mathsf{H}} \mathbf{d}_j \rangle\right|^2}{\|\mathbf{\Gamma}_{\mathbf{W}}^{-1/2} \mathbf{W}^{\mathsf{H}} \mathbf{d}_i\|^2 \|\mathbf{\Gamma}_{\mathbf{W}}^{-1/2} \mathbf{W}^{\mathsf{H}} \mathbf{d}_j\|^2}$$

where $\mathbf{\Gamma}_{\mathbf{W}} = \mathrm{diag}(\|\mathbf{w}_1\|^2, \ldots, \|\mathbf{w}_T\|^2)$

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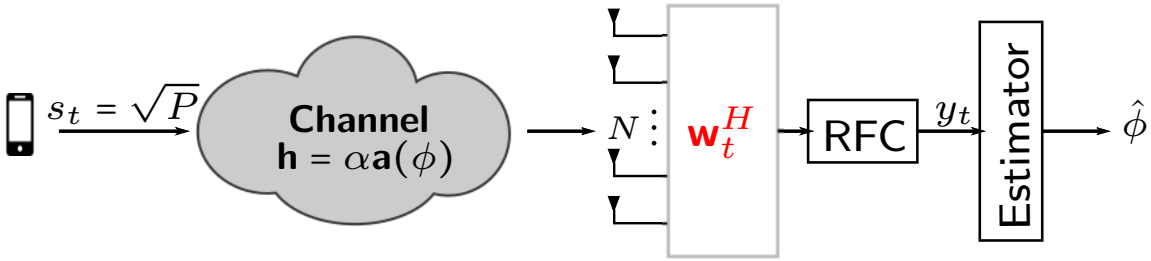
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2. Writing $\mathbf{R} = \mathbf{W} \mathbf{\Gamma}_{\mathbf{W}}^{-1} \mathbf{W}^{\mathsf{H}}$, and using epigraph formulation,

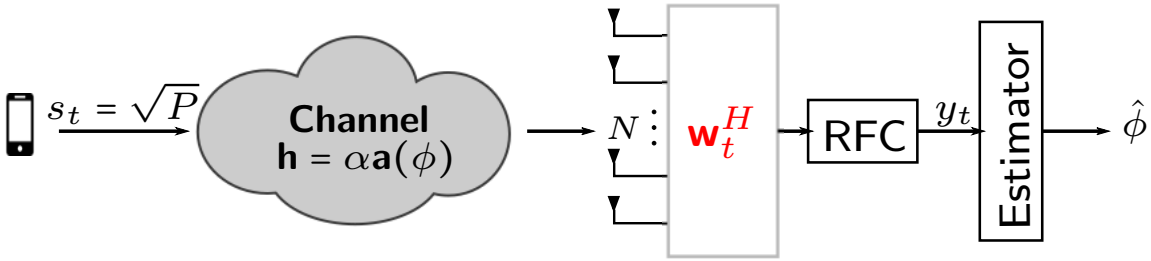
$$\begin{aligned} \underset{\mathbf{R}, \gamma}{\text{minimize}} \quad & \gamma \\ \text{subject to} \quad & \begin{bmatrix} \sqrt{\gamma} \mathbf{d}_i^{\mathsf{H}} \mathbf{R} \mathbf{d}_i & \mathbf{d}_i^{\mathsf{H}} \mathbf{R} \mathbf{d}_j \\ \mathbf{d}_j^{\mathsf{H}} \mathbf{R} \mathbf{d}_i & \sqrt{\gamma} \mathbf{d}_j^{\mathsf{H}} \mathbf{R} \mathbf{d}_j \end{bmatrix} \succeq 0, \quad \forall\, i \neq j, \\ & \mathrm{tr}(\mathbf{R}) = T, \quad \mathbf{R} \succeq 0, \\ & \mathrm{rank}(\mathbf{R}) \leq T \end{aligned}$$

   $\Rightarrow$ Rank-constrained SDP that can be approached using convex iteration method

Numerical Example: Angle-of-Arrival (AoA) Estimation

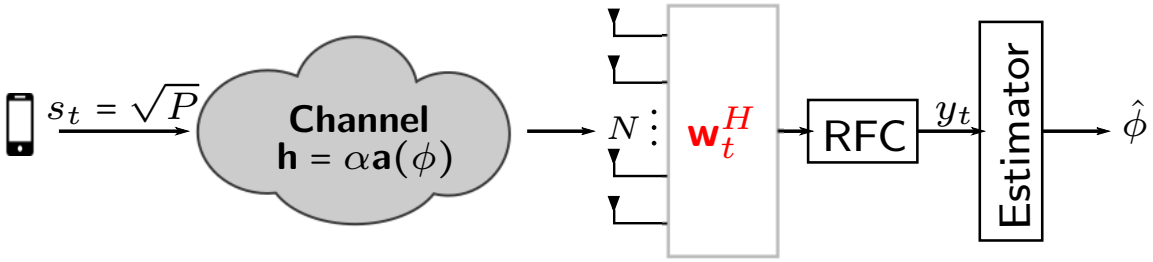


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- BS with N antennas and a single RF chain
- UE sends pilot symbols, BS estimates the angle of arrival (AoA) ϕ
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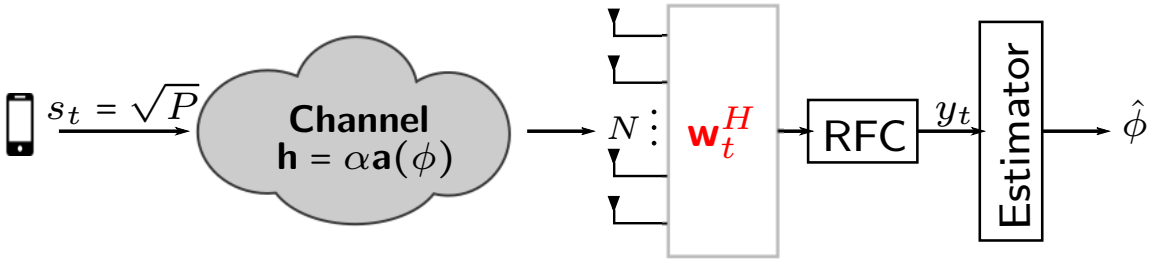
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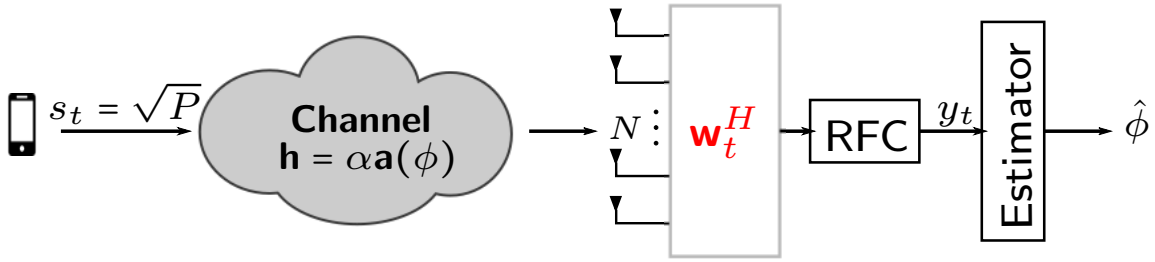
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- Measurement model:

$$\mathbf{y} = \alpha \sqrt{P} \mathbf{W}^{\mathsf{H}} \mathbf{D} \mathbf{u} + \mathbf{z}$$

where $\mathbf{u}$ is 1-sparse, and $\mathbf{D} = \begin{bmatrix} \mathbf{a}(\theta_1) & \cdots & \mathbf{a}(\theta_M) \end{bmatrix} \in \mathbb{C}^{N \times M}$.

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Question: Given array response $\mathbf{D}$, how to design the beamforming matrix $\mathbf{W}$?

Existing Sensing Strategies for AoA Estimation

- Beam sweeping: Set $\mathbf{w}_t = \mathbf{a}(\theta_t),\ \forall\, t = 1, \ldots, M$

[4] W. Huleihel, J. Tabrikian, and R. Shavit. "Optimal Adaptive Waveform Design for Cognitive MIMO Radar". In: *IEEE Trans. Signal Process.* 61.20 (2013), pp. 5075–5089.

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- Bayesian Cramér-Rao bound optimization[4]
- Designs based on channel codes:
  1. Codebook matching[5]: Design $\mathbf{W}$ such that $\mathbf{W}^{\mathsf{H}}\mathbf{D} = \mathbf{C}$ for a given codebook $\mathbf{C}$ $\Rightarrow$ Works if $N \geq M$
  2. Golomb-ruler-based construction[6]: Works if $M = N = T^2 - 1$

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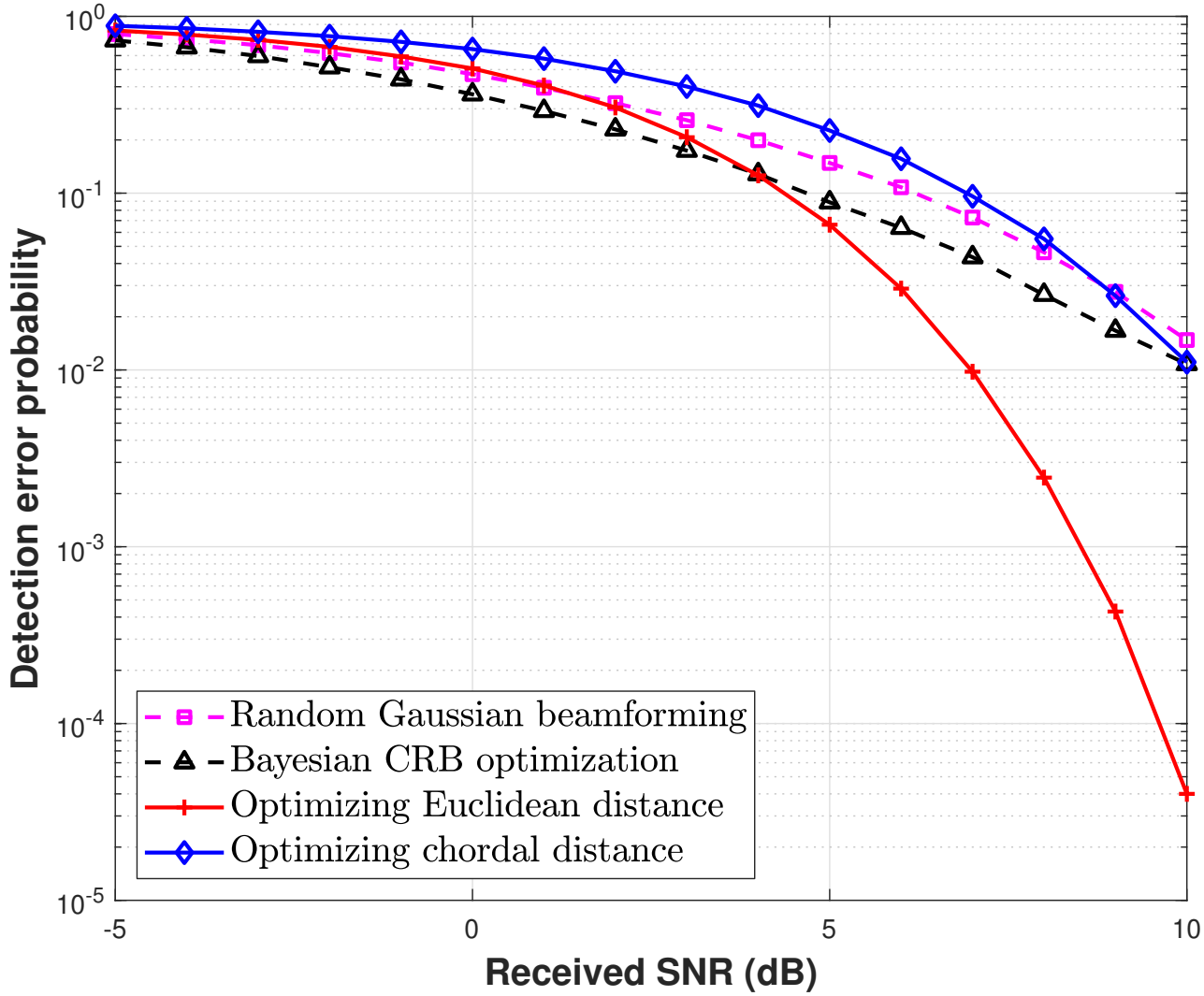
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Simulation Results - Known α

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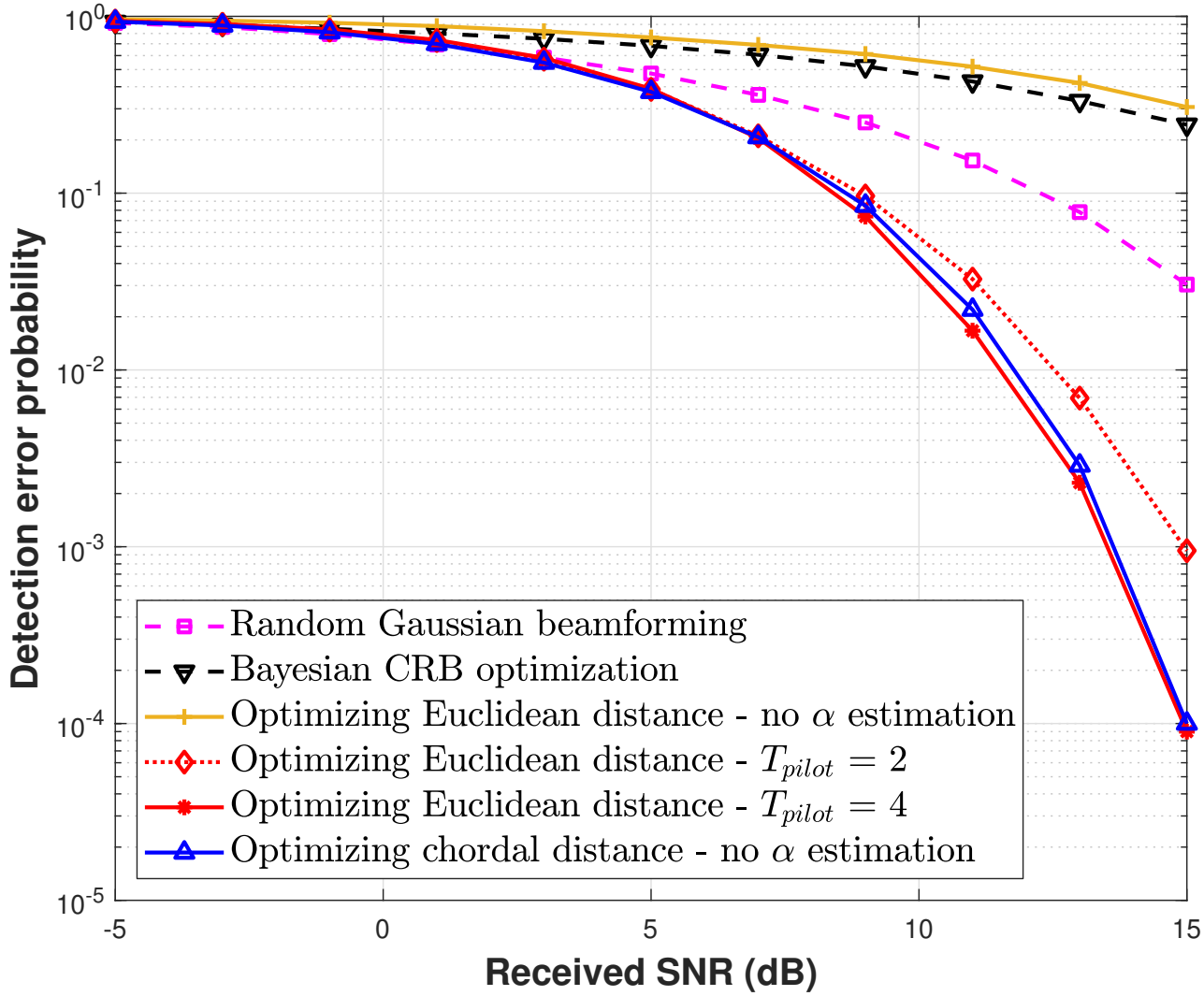


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Concluding Remarks

- Sensing matrix design for 1-sparse problems:
  1. Known α: Maximize the minimum ℓ_2 distance between codewords
  2. Unknown α: Maximize the minimum chordal distance between codewords

- Designs account for the structure of the dictionary matrix

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- **Future directions:**
  1. Sensing matrix design for k-sparse problems
  2. Alternative performance metrics (e.g., MSE)